

# Rank-3 root systems induce root systems of rank 4 via a new Clifford spinor construction

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**Abstract.** In this paper, we show that via a novel construction every rank-3 root system induces a root system of rank 4. In a Clifford algebra framework, an even number of successive Coxeter reflections yields – via the Cartan-Dieudonné theorem – spinors that describe rotations. In three dimensions these spinors themselves have a natural four-dimensional Euclidean structure, and discrete spinor groups can therefore be interpreted as 4D polytopes. In fact, these polytopes have to be root systems, thereby inducing Coxeter groups of rank 4. For the corresponding case in two dimensions, the groups  $I_2(n)$  are shown to be self-dual.

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## 1. Introduction

Root systems are useful mathematical abstractions, which are polytopes that generate reflection (Coxeter) groups. Certain families of root systems exist in any dimension, whereas others – exceptional ones – only exist as accidental structures in specific dimensions. Root systems in different dimensions are largely thought to be independent of each other (with the exception of sub-root systems). The Freudenthal-Tits magic square [1] makes some non-trivial connections, but geometric insight as to why these should exist is scarce. In this paper, we present a novel connection between root systems in different dimensions that has a geometric origin.

Clifford's Geometric Algebra provides a mathematical framework that generalises the more familiar vector space and matrix methods. In this setup, orthogonal transformations are encoded (in fact, doubly covered) by versors, which are the *geometric product* of several unit vectors, via a *sandwiching prescription*. In particular, the rotations (i.e. the special orthogonal group) are doubly covered by geometric products of an even number of unit vectors, resulting in *spinors*, or *rotors*. These elements in the even-subalgebra can themselves have a Euclidean structure and can

thus be reinterpreted as vectors in a different space. Here, we show that such a construction can induce a root system from a given root system. This systematises the observations made in [3, 4, 5], and opens up a new – spinorial – view of the geometry of root systems.

The rest of this paper is organised as follows. Section 2 introduces Coxeter groups and root systems, and Section 3 gives the necessary Clifford algebra background. The Induction Theorem is stated and proved in Section 4, and the self-duality of the two-dimensional case is discussed in Section 5. We conclude in Section 6.

## 2. Coxeter Groups

**Definition 2.1 (Coxeter group).** A Coxeter group is a group generated by some involutive generators  $s_i, s_j \in S$  subject to relations of the form  $(s_i s_j)^{m_{ij}} = 1$  with  $m_{ij} = m_{ji} \geq 2$  for  $i \neq j$ .

The finite Coxeter groups have a geometric representation where the involutions are realised as reflections at hyperplanes through the origin in a Euclidean vector space  $\mathcal{E}$  (essentially just the classical reflection groups). In particular, let  $(\cdot|\cdot)$  denote the inner product in  $\mathcal{E}$ , and  $\lambda, \alpha \in \mathcal{E}$ .

**Definition 2.2 (Reflections and roots).** The generator  $s_\alpha$  corresponds to the reflection

$$s_\alpha : \lambda \rightarrow s_\alpha(\lambda) = \lambda - 2 \frac{(\lambda|\alpha)}{(\alpha|\alpha)} \alpha \quad (2.1)$$

at a hyperplane perpendicular to the root vector  $\alpha$ .

The action of the Coxeter group is to permute these root vectors, and its structure is thus encoded in the collection  $\Phi \in \mathcal{E}$  of all such roots, which form a root system:

**Definition 2.3 (Root system).** Root systems are defined by the two axioms

1.  $\Phi$  only contains a root  $\alpha$  and its negative, but no other scalar multiples:  $\Phi \cap \mathbb{R}\alpha = \{-\alpha, \alpha\} \ \forall \alpha \in \Phi$ .
2.  $\Phi$  is invariant under all reflections corresponding to vectors in  $\Phi$ :  $s_\alpha \Phi = \Phi \ \forall \alpha \in \Phi$ .

## 3. Geometric Algebra

The study of Clifford algebras and Geometric Algebra originated with Grassmann's, Hamilton's and Clifford's geometric work [7, 8, 2]. However, the geometric content of the algebras was soon lost when interesting algebraic properties were discovered in mathematics, and Gibbs advocated the use of vector calculus and quaternions in physics. When Clifford algebras resurfaced in physics in the context of quantum mechanics, it was purely for their algebraic properties, and this continues in particle physics to this day. Thus, it is widely thought that Clifford algebras are somehow

intrinsically quantum mechanical in nature. The original geometric meaning of Clifford algebras has been revived in the work of David Hestenes [10, 9, 11]. Here, we follow an exposition along the lines of [6].

In a manner reminiscent of complex numbers carrying both real and imaginary parts in the same algebraic entity, one can consider the geometric product of two vectors defined as the sum of their scalar (inner/symmetric) product and wedge (outer/exterior/antisymmetric) product

$$ab \equiv a \cdot b + a \wedge b. \quad (3.1)$$

The wedge product is the outer product introduced by Grassmann, as an antisymmetric product of two vectors, which naturally defines a plane. Unlike the constituent inner and outer products, the geometric product is invertible, as  $a^{-1}$  is simply given by  $a^{-1} = a/(a^2)$ . This leads to many algebraic simplifications over standard vector space techniques, and also feeds through to the differential structure of the theory, with Green's function methods that are not achievable with vector calculus methods. This geometric product can be extended to the product of more vectors via associativity and distributivity, resulting in higher grade objects called multivectors. There are a total of  $2^n$  elements in the algebra, since it truncates at grade  $n$  multivectors due to the scalar nature of the product of parallel vectors and the antisymmetry of orthogonal vectors. Essentially, a Clifford algebra is a deformation of the exterior algebra by a quadratic form, and for a Geometric Algebra this is the metric of space(time).

The geometric product provides a very compact and efficient way of handling reflections in any number of dimensions, and thus by the Cartan-Dieudonné theorem also rotations. For a unit vector  $n$ , we consider the reflection of a vector  $a$  in the hyperplane orthogonal to  $n$ . Thanks to the geometric product, in Clifford algebra the two terms in Eq. (2.1) combine into a single term, and thus a 'sandwiching prescription':

**Theorem 3.1 (Reflections).** *In Geometric Algebra, a vector  $a$  transforms under a reflection in the (hyper-)plane defined by a unit normal vector  $n$  as*

$$a' = -nan. \quad (3.2)$$

This is a remarkably compact and simple prescription for reflecting vectors in hyperplanes. More generally, higher grade multivectors transform similarly ('co-variantly'), as  $M = ab \dots c \rightarrow \pm nannbn \dots ncn = \pm nab \dots cn = \pm nMn$ . Even more importantly, from the Cartan-Dieudonné theorem, rotations are the product of successive reflections. For instance, compounding the reflections in the hyperplanes defined by the unit vectors  $n$  and  $m$  results in a rotation in the plane defined by  $n \wedge m$ .

**Proposition 3.2 (Rotations).** *In Geometric Algebra, a vector  $a$  transforms under a rotation in the plane defined by  $n \wedge m$  via successive reflection in hyperplanes determined by the unit vectors  $n$  and  $m$  as*

$$a'' = mnanm =: Ra\tilde{R}, \quad (3.3)$$

where we have defined  $R = mn$  and the tilde denotes the reversal of the order of the constituent vectors  $\tilde{R} = nm$ .

**Theorem 3.3 (Rotors and spinors).** *The object  $R = mn$  generating the rotation in Eq. (3.3) is called a rotor. It satisfies  $\tilde{R}R = R\tilde{R} = 1$ . Rotors themselves transform single-sidedly under further rotations, and thus form a multiplicative group under the geometric product, called the rotor group. Since  $R$  and  $-R$  encode the same rotation, the rotor group is a double-cover of the special orthogonal group, and thus essentially the Spin group. Objects in Geometric Algebra that transform single-sidedly are called spinors, so that rotors are normalised spinors.*

**Corollary 3.4 (Discrete spinor groups).** *Discrete spinor groups are of even order.*

Higher multivectors transform in the above covariant, double-sided way as  $MN \rightarrow (RM\tilde{R})(RN\tilde{R}) = RM\tilde{R}RN\tilde{R} = R(MN)\tilde{R}$ .

The Geometric Algebra of three dimensions  $\text{Cl}(3)$  spanned by three orthogonal unit vectors  $e_1, e_2$  and  $e_3$  contains three bivectors  $e_1e_2, e_2e_3$  and  $e_3e_1$  that square to  $-1$ , as well as the highest grade object  $e_1e_2e_3$  (trivector and pseudoscalar), which also squares to  $-1$ .

$$\underbrace{\{1\}}_{1 \text{ scalar}} \quad \underbrace{\{e_1, e_2, e_3\}}_{3 \text{ vectors}} \quad \underbrace{\{e_1e_2 = Ie_3, e_2e_3 = Ie_1, e_3e_1 = Ie_2\}}_{3 \text{ bivectors}} \quad \underbrace{\{I \equiv e_1e_2e_3\}}_{1 \text{ trivector}}. \quad (3.4)$$

**Theorem 3.5 (Quaternions and spinors of  $\text{Cl}(3)$ ).** *The unit spinors  $\{1; Ie_1; Ie_2; Ie_3\}$  of  $\text{Cl}(3)$  are isomorphic to the quaternion algebra  $\mathbb{H}$ .*

This completes the background that we shall need for our proof of the Induction Theorem.

## 4. Induction Theorem

In this section, we show that every root system of rank 3 induces a root system in dimension 4.

**Proposition 4.1 ( $O(4)$ -structure of spinors and quaternions).** *The space of  $\text{Cl}(3)$ -spinors and quaternions have a 4D Euclidean signature.*

*Proof.* For quaternions, this is given via conjugation defined by  $\bar{q} = q_0 - q_ie_i$ , as  $(p, q) = \frac{1}{2}(\bar{p}q + p\bar{q})$ ,  $|q|^2 = \bar{q}q = q_0^2 + q_1^2 + q_2^2 + q_3^2$ . For a spinor  $\psi = a_0 + a_1Ie_1 + a_2Ie_2 + a_3Ie_3$ , it is given by  $\psi\tilde{\psi} = a_0^2 + a_1^2 + a_2^2 + a_3^2$  (or via Theorem 3.5).  $\square$

**Lemma 4.2 (Discrete quaternion groups give root systems).** *Any finite subgroup  $G$  of even order in  $\mathbb{H}$  is a root system.*

*Proof.* This is stated and proven in [12].  $\square$

**Lemma 4.3 (Rank-3 Coxeter groups and finite even quaternion groups).** *The spinors defined from any rank-3 Coxeter group are isomorphic to an even subgroup of the quaternions.*

*Proof.* From Corollary 3.4, the spinor group generated by a Coxeter group is discrete and even. Because of Theorem 3.5, for a Coxeter group of rank 3 this spinor group is isomorphic to a finite even order quaternion group.  $\square$

**Lemma 4.4 (Discrete spinor groups in 3D give 4D root systems).** *A discrete group of spinors in three dimensions is a four-dimensional root system.*

*Proof.* Due to Lemma 4.3, the discrete spinor group is even and isomorphic to an even quaternion group. From Lemma 4.2 it follows that this is a root system.  $\square$

**Theorem 4.5 (Induction Theorem).** *Any rank-3 root system induces a root system of rank 4.*

*Proof.* A root system in three dimensions gives rise to corresponding Coxeter reflections (Section 2), acting in Geometric Algebra as given by Eq. (3.2). An even number of successive reflections yields spinors via Theorem 3.3, and from Corollary 3.4, this group is even. Via Lemma 4.4, this even spinor group yields a root system in four dimensions.  $\square$

**Theorem 4.6 (Non-existence of a reduction theorem).** *Not every rank-4 root system can be induced by a rank-3 root system.*

*Proof.* A counterexample is provided by  $I_2(4) \times A_1 \times A_1$ .  $\square$

*Example.* The simple roots of  $A_1 \times A_1 \times A_1$  can be taken as  $\alpha_1 = e_1$ ,  $\alpha_2 = e_2$  and  $\alpha_3 = e_3$ . Closure of these under reflections via Eq. (3.2) gives  $(\pm 1, 0, 0)$  and permutations thereof, which are the 6 vertices of the root system, the octahedron. Combining two reflections yields a spinor, so forming rotors according to  $R_{ij} = \alpha_i \alpha_j$  gives, e.g.  $R_{11} = \alpha_1^2 = 1 \equiv (1; 0, 0, 0)$ , or  $R_{23} = \alpha_2 \alpha_3 = e_2 e_3 = I e_1 \equiv (0; 1, 0, 0)$ . Explicit calculation of all cases generates the 8 permutations of  $(\pm 1; 0, 0, 0)$ . When interpreted as a 4D polytope, these are the vertices of the 16-cell, which is the root system of  $A_1 \times A_1 \times A_1 \times A_1$ .

Other examples are  $A_3 \rightarrow D_4$  (cuboctahedron to 24-cell),  $B_3 \rightarrow F_4$  and  $H_3 \rightarrow H_4$  (icosidodecahedron to 600-cell) [3, 4, 5].

## 5. Spinors in dimension two

The space of spinors  $\psi = a + be_1e_2 \equiv a + bI$  in two-dimensional Euclidean space is also two-dimensional, and has a natural Euclidean structure given by  $\psi\tilde{\psi} = a^2 + b^2$ . This induces a rank-2 root system from any rank-2 root system in a similar way to the construction above. However, this construction does not yield any new root systems by the following theorem.

**Theorem 5.1 (Self-duality of  $I_2(n)$ ).** *Two-dimensional root systems are self-dual under the Clifford spinor construction.*

*Proof.* A 2D root vector  $\alpha_i = a_1e_1 + a_2e_2$  is in bijection with a spinor by  $\alpha_i \rightarrow \alpha_1 \alpha_i = e_1 \alpha_i = a_1 + a_2e_1e_2 = a_1 + a_2I$  (taking  $\alpha_1 = e_1$  without loss of generality). This is the same as forming a spinor between those two root vectors. The infinite family of two-dimensional root systems  $I_2(n)$  is therefore self-dual. For instance, with  $n = 2$ , the order of the Coxeter group  $|W|$  matches the number of roots  $|\Phi|$  for the following exemplary cases: for  $A_2$  one has  $(n+1)! = n(n+1)$ , for  $B_2$ ,  $2^n n! = 2n^2$  holds, and for  $G_2$   $12 = 12$ .  $\square$

## 6. Conclusions

We have shown how via a Clifford spinor construction, any root system of rank 3 induces a root system in four dimensions. In the two-dimensional case, root systems (i.e.  $I_2(n)$ ) were shown to be self-dual. This spinorial view sheds light on the peculiarities of root systems, in particular certain rank-4 root systems (see [3]), and more generally, opens up a new field of study in the geometry of root systems.

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